

Polynomial splittings of correction terms and doubly slice knots

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Abstract. We show that if the connected sum of two knots with coprime Alexander polynomials is doubly slice, then the Ozsváth–Szabó correction terms as smooth double sliceness obstructions vanish for both knots. Recently, Jeffrey Meier gave smoothly slice knots that are topologically doubly slice, but not smoothly doubly slice. As an application, we give a new example of such knots that is distinct from Meier’s knots modulo doubly slice knots.

1. Introduction

A knot K in the 3-sphere S^3 is *doubly slice* if there is a smoothly embedded and unknotted 2-sphere S in the 4-sphere S^4 which transversely intersects the standard S^3 in S^4 at K . If we allow S to be a topologically locally flat embedded and unknotted 2-sphere, then K is called *topologically doubly slice*. A knot is *slice* if it bounds a smoothly embedded disk in the 4-ball D^4 . Obviously a doubly slice knot is slice.

There have been known results on splittings of sliceness obstructions for knots with coprime Alexander polynomials. Let K_1 and K_2 be knots and $K := K_1 \# K_2$, their connected sum. Suppose K_1 and K_2 have coprime Alexander polynomials. Under this hypothesis, Levine [Lev69] showed that if K is algebraically slice, then all K_i are algebraically slice, giving $p(t)$ -primary decomposition of the algebraic concordance group. The first author [Kim05] showed that if K has vanishing Casson–Gordon invariants, then so do all K_i . The authors [KK08] showed that if K has vanishing metabelian von Neumann–Cheeger–Gromov $\rho^{(2)}$ -invariants, then so do all K_i . Later the authors [KK14] extended the result in [KK08] using higher-order $\rho^{(2)}$ -invariants, and gave a knot which has vanishing Casson–Gordon invariants and has concordance genus greater than 1. We note that Cochran–Harvey–Leidy’s work in [CHL11] gave evidence of the $p(t)$ -primary decomposition of the solvable filtration of Cochran–Orr–Teichner in [COT03] which would extend Levine’s $p(t)$ -primary decomposition of the algebraic concordance group in [Lev69].

Similar results are known for d -invariants of Ozsváth–Szabó. For a rational homology 3-sphere Y and $\text{Spin}^c(Y)$, the set of Spin^c structures of Y , Ozsváth–Szabó [OS03] defined a function $d: \text{Spin}^c(Y) \rightarrow \mathbb{Q}$, which is called the *correction term*. Via prime power fold branched cyclic covers of S^3 over a knot, the d -invariants give rise to sliceness obstructions (see Theorem 2.4).

For a knot K and a prime power $q = p^r$, we denote by $\Sigma^q(K)$ the q -fold branched cover of S^3 over K . It is known that $H_1(\Sigma^q(K))$ acts freely and transitively on $\text{Spin}^c(\Sigma^q(K))$. We denote by $\mathfrak{s} + a$ the element obtained by the action of $a \in H_1(\Sigma^q(K))$ on $\mathfrak{s} \in \text{Spin}^c(\Sigma^q(K))$. Let $\mathfrak{s}_0 \in \text{Spin}^c(\Sigma^q(K))$ be the *canonical* Spin^c structure of (K, q) (see Section 2.2), and for each $a \in H_1(\Sigma^q(K))$ define

$$\bar{d}(\Sigma^q(K), \mathfrak{s}_0 + a) := d(\Sigma^q(K), \mathfrak{s}_0 + a) - d(\Sigma^q(K), \mathfrak{s}_0).$$

We say that a knot K has *vanishing d -invariants on $\Sigma^q(K)$* if there exists a metabolizer G for the linking form $H_1(\Sigma^q(K)) \times H_1(\Sigma^q(K)) \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $\bar{d}(\Sigma^q(K), \mathfrak{s}_0 + a) = 0$ for all $a \in G$.

2010 *Mathematics Subject Classification.* 57M25 (primary), 57N70 (secondary) .

Key words and phrases. Doubly slice knot, Alexander polynomial, Correction term, d -invariant.

Hedden–Livingston–Ruberman [HLR12, Theorem 3.2] showed that if $H_1(\Sigma^q(K_1))$ and $H_1(\Sigma^q(K_2))$ have coprime orders and $K_1 \# K_2$ is slice, then each K_i has vanishing d -invariants on $\Sigma^q(K_i)$. Bao [Bao15, Theorem 1.1] showed that if K_1 and K_2 have coprime Alexander polynomials and $K_1 \# K_2$ is slice, then each K_i has vanishing d -invariants on $\Sigma^{p^r}(K_i)$ for all but finitely many primes p and all natural numbers r . She also showed that the finiteness restriction on p can be removed if $K_1 \# K_2$ is ribbon.

We present a new splitting theorem of d -invariants concerning double sliceness, and give an application.

Definition 1.1. Given a prime power q , a knot K has *doubly vanishing d -invariants on $\Sigma^q(K)$* if there exist metabolizers G_1 and G_2 for the linking form $H_1(\Sigma^q(K)) \times H_1(\Sigma^q(K)) \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $H_1(\Sigma^q(K)) = G_1 \oplus G_2$ and $\bar{d}(\Sigma^q(K), \mathfrak{s}_0 + a) = 0$ for all $a \in G_1 \cup G_2$. A knot is said to have *doubly vanishing d -invariants* if it has doubly vanishing d -invariants on $\Sigma^q(K)$ for every prime power q .

It is well-known to the experts and has appeared in various forms in the literature that a doubly slice knot has doubly vanishing d -invariants (for example, see Theorem 2.5). We give the following new splitting theorem of d -invariants:

Theorem 1.2 (Main Theorem). *Let K_1 and K_2 be knots. Suppose that the Alexander polynomials of K_1 and K_2 are coprime in $\mathbb{Q}[t^{\pm 1}]$. If $K_1 \# K_2$ is doubly slice, then both K_1 and K_2 have doubly vanishing d -invariants.*

We note that there are knots K_1 and K_2 such that their Alexander polynomials are coprime, but $H_1(\Sigma^q(K_1))$ and $H_1(\Sigma^q(K_2))$ have the same order for all prime power q [Kim09]. For these knots the previous theorem is necessary to see that d -invariants split. Theorem 1.2 can be considered as an extension of [Bao15, Theorem 1.1] to the case of double sliceness, and can be proved using Seifert forms similarly as done by Bao for the case of sliceness. However we prove it using Blanchfield forms, which, we think, is a simpler way.

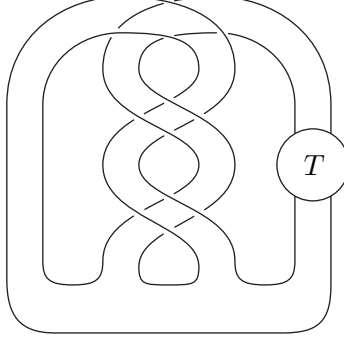
Unlike Bao’s theorem, we do not need any finiteness condition on primes p for double sliceness. A crucial distinction between doubly slice knots and slice knots is that a doubly slice knot bounds two slice disks D_{\pm} such that $D_+ \cup D_-$ is an unknotted 2-sphere in S^4 , and therefore $H_1(D^4 \setminus D_{\pm}; \mathbb{Z}[t^{\pm 1}])$ are \mathbb{Z} -torsion free. See the proof of Theorem 1.2 for details.

Meier [Mei15] gave an infinite family of slice knots K_p (denoted $\mathcal{K}_{p,k}$ in [Mei15]) for odd primes p that are topologically doubly slice, but not doubly slice. As an application of our main theorem, we give another example of a slice knot K distinct from all K_p , modulo double sliceness, that is topologically doubly slice, but not doubly slice. Explicitly, let T be the positive untwisted Whitehead double of the right-handed trefoil and let K be the knot given in Figure 1. Namely, K is a satellite of T with pattern the 9_{46} knot. We have the following theorem.

Theorem 1.3. *Let K be the knot in Figure 1. Let $n, n_i \in \mathbb{Z}$ and let p_i be odd primes.*

- (1) *The knot K is slice, topologically doubly slice, but not doubly slice. Indeed, K does not have doubly vanishing d -invariants..*
- (2) *Suppose $n = 1$ or $n_i = 1$ for some i . Then, the knot $nK \# (\#_{i=1}^m n_i K_{p_i})$ is not doubly slice. In particular, $K \# (-K_{p_i})$ is not doubly slice for all p_i .*

This paper is organized as follows. We give the background material on Blanchfield forms, linking forms, and the correction terms in Section 2. We prove Theorems 1.2 and 1.3 in Section 3.

FIGURE 1. The knot K .

Acknowledgments

The first author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2011-0012893). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (no. 2011-0030044(SRC-GAIA) and no. 2015R1D1A1A01056634).

2. Preliminaries

2.1. Blanchfield forms and linking forms

For a knot K , let $M(K)$ denote the zero framed surgery on K in S^3 . Let $\Lambda := \mathbb{Z}[t^{\pm 1}]$. It is known that $H_1(S^3 \setminus K; \Lambda) \cong H_1(M(K); \Lambda)$ as Λ -modules, and there is a nonsingular hermitian sesquilinear form

$$Bl: H_1(M(K); \Lambda) \times H_1(M(K); \Lambda) \longrightarrow S^{-1}\Lambda/\Lambda,$$

which is called the *Blanchfield form* of K . Here $S^{-1} = \{f \in \Lambda \mid f(1) = 1\}$. For a Λ -submodule P of $H_1(M(K); \Lambda)$, let

$$P^\perp := \{y \in H_1(M(K); \Lambda) \mid Bl(x, y) = 0 \text{ for all } x \in P\}.$$

We say that P is a *metabolizer* for the Blanchfield form if $P = P^\perp$.

For a disk D properly embedded in D^4 , let $X(D) := D^4 \setminus N(D)$ where $N(D)$ is the open tubular neighborhood of D . If D is a slice disk for a knot K , then $\partial X(D) = M(K)$. We have the following well-known proposition, for example see [Fri03, Proposition 2.7].

Proposition 2.1. *Let K be a slice knot and D a slice disk for K in D^4 . Suppose $H_1(X(D); \Lambda)$ is \mathbb{Z} -torsion free. Then $\text{Ker}\{i_*: H_1(M(K); \Lambda) \rightarrow H_1(X(D); \Lambda)\}$ is a metabolizer for the Blanchfield form, where i_* is the homomorphism induced from the inclusion.*

Recall that for a prime power q we let $\Sigma^q(K)$ denote the q -fold branched cyclic cover of S^3 over K . Then $\Sigma^q(K)$ is a rational homology 3-sphere and there is a nonsingular linking form $\lambda^q: H_1(\Sigma^q(K)) \times H_1(\Sigma^q(K)) \rightarrow \mathbb{Q}/\mathbb{Z}$. For a subgroup P of $H_1(\Sigma^q(K))$, we define

$$P^\perp := \{y \in H_1(\Sigma^q(K)) \mid \lambda^q(x, y) = 0 \text{ for all } x \in P\}.$$

We say that P is a *metabolizer* for λ^q if $P = P^\perp$. If P is a metabolizer, we have $|P|^2 = |H_1(\Sigma^q(K))|$. It is known that $H_1(\Sigma^q(K)) = H_1(M(K); \Lambda)/(t^q - 1)$ as Λ -modules,

and we say that P is a Λ -metabolizer if $P = P^\perp$ and P is a Λ -submodule of $H_1(\Sigma^q(K))$. For a disk D properly embedded in D^4 , let $W^q(D)$ be the q -fold branched cyclic cover of D^4 over D . If a knot bounds a slice disk D in D^4 , then $W^q(D)$ is a rational homology 4-ball such that $\partial W^q(D) = \Sigma^q(K)$. We have the following well-known proposition, for example see [Fri03, Proposition 2.15].

Proposition 2.2. *Let K be a slice knot and D a slice disk for K in D^4 . Then $\text{Ker}\{H_1(\Sigma^q(K)) \rightarrow H_1(W^q(D))\}$ is a Λ -metabolizer for the linking form λ^q .*

We relate Blanchfield forms to linking forms. Define

$$\pi^q: H_1(M(K); \Lambda) \longrightarrow H_1(M(K); \Lambda)/(t^q - 1) = H_1(\Sigma^q(K))$$

to be the projection map.

Proposition 2.3 ([Fri03, Proposition 2.18]). *Let K be a knot. If $P \subset H_1(M(K); \Lambda)$ is a metabolizer for the Blanchfield form on $H_1(M(K); \Lambda)$, then the submodule $\pi^q(P) \subset H_1(\Sigma^q(K))$ is a Λ -metabolizer for the linking form λ^q on $H_1(\Sigma^q(K))$.*

2.2. Spin^c structures and correction terms

Let $\mathfrak{s}_0 \in \text{Spin}^c(\Sigma^q(K))$ be the *canonical* Spin^c structure of (K, q) defined as follows: let $f: \Sigma^q(K) \rightarrow S^3$ be the branched covering map and let $K' := f^{-1}(K)$. Then \mathfrak{s}_0 is defined to be the unique Spin^c structure whose restriction to $\Sigma^q(K) \setminus N(K')$ is the pull-back $f^*(\mathfrak{s})$ of the unique Spin^c structure \mathfrak{s} on $S^3 \setminus N(K)$. Here $N(K')$ and $N(K)$ denote the open tubular neighborhoods of K' and K , respectively. We note that the Spin^c structure \mathfrak{s}_0 is equal to the Spin^c structure given in [GRS08, Lemma 2.1] (see [Jab12, Remark 2.5]). For more details, refer to [Jab12, Section 2]. Now using \mathfrak{s}_0 we can identify $H_1(\Sigma^q(K))$ with $\text{Spin}^c(\Sigma^q(K))$ via the map $a \mapsto \mathfrak{s}_0 + a$ for $a \in H_1(\Sigma^q(K))$.

With the canonical Spin^c structure $\mathfrak{s}_0 \in \text{Spin}^c(\Sigma^q(K))$, we have the following sliceness obstruction.

Theorem 2.4 ([GRS08]). *Let q be a prime power. Suppose that a knot K bounds a slice disk D in D^4 . Then, $d(\Sigma^q(K), \mathfrak{s}_0 + a) = 0$ for all $a \in \text{Ker}\{i_*: H_1(\Sigma^q(K)) \rightarrow H_1(W^q(D))\}$ where i_* is the homomorphism induced from the inclusion. In particular, if K is slice, then $d(\Sigma^q(K), \mathfrak{s}_0) = 0$ and moreover there is a Λ -metabolizer $P \subset H_1(\Sigma^q(K))$ such that $d(\Sigma^q(K), \mathfrak{s}_0 + a) = 0$ for all $a \in P$.*

As obstructions for a knot to being doubly slice, we have the following theorem.

Theorem 2.5 ([Mei15, Theorem 2.2]). *Let q be a prime power. If K is a doubly slice knot, then $H_1(\Sigma^q(K)) = M_+ \oplus M_-$ where M_\pm are Λ -metabolizers for the linking form λ^q on $H_1(\Sigma^q(K))$ and $d(\Sigma^q(K), \mathfrak{s}_0 + a) = 0$ for all $a \in M_+ \cup M_-$. In particular, K has doubly vanishing d -invariants.*

3. Proofs of Theorems 1.2 and 1.3

First we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let q be a prime power, and $K := K_1 \# K_2$. Since K is doubly slice, $K = S \cap S^3$ where S is an unknotted 2-sphere in S^4 which transversely intersects S^3 , where S^3 is the standard 3-sphere in S^4 . We regard $S^4 = D_+^4 \cup D_-^4$, a union of two 4-balls, such that $\partial D_\pm^4 = S^3$. Let $D_+ := S \cap D_+^4$ and $D_- := S \cap D_-^4$.

Since S is unknotted in S^4 , using a Mayer-Vietoris sequence, we obtain

$$H_1(S^3 \setminus K; \Lambda) = H_1(D_+^4 \setminus D_+; \Lambda) \oplus H_1(D_-^4 \setminus D_-; \Lambda).$$

Since $H_1(S^3 \setminus K; \Lambda)$ is \mathbb{Z} -torsion free, the summands on the right hand side are \mathbb{Z} -torsion free. Therefore, letting

$$M_{\pm} := \text{Ker}\{i_{\pm}: H_1(S^3 \setminus K; \Lambda) \longrightarrow H_1(D_{\pm}^4 \setminus D_{\pm}; \Lambda)\}$$

where i_{\pm} are homomorphisms induced from inclusions, by Proposition 2.1 the M_{\pm} are metabolizers for the Blanchfield form

$$Bl: H_1(S^3 \setminus K; \Lambda) \times H_1(S^3 \setminus K; \Lambda) \longrightarrow S^{-1}\Lambda/\Lambda$$

and $H_1(S^3 \setminus K; \Lambda) = M_+ \oplus M_-$.

Note that $H_1(S^3 \setminus K; \Lambda) = H_1(S^3 \setminus K_1; \Lambda) \oplus H_1(S^3 \setminus K_2; \Lambda)$. For $i = 1, 2$, let

$$M_{\pm}^i := M_{\pm} \cap H_1(S^3 \setminus K_i; \Lambda).$$

Then clearly $M_{\pm}^1 \oplus M_{\pm}^2 \subset M_{\pm}$. We need the following lemma, of which proof will be given later.

Lemma 3.1. $M_{\pm} = M_{\pm}^1 \oplus M_{\pm}^2$ and M_{\pm}^i are metabolizers for the Blanchfield form on $H_1(S^3 \setminus K_i; \Lambda)$ for $i = 1, 2$.

Let W_+ (resp. W_-) be the q -fold cyclic cover of D_+^4 (resp. D_-^4) branched over D_+ (resp. D_-). Then $\partial W_{\pm} = \Sigma^q(K)$. Note that

$$\begin{aligned} H_1(\Sigma^q(K)) &\cong H_1(S^3 \setminus K; \Lambda)/(t^q - 1), \\ H_1(W_{\pm}) &\cong H_1(D_{\pm}^4 \setminus D_{\pm}; \Lambda)/(t^q - 1). \end{aligned}$$

Therefore we have the following commutative diagram:

$$\begin{array}{ccc} H_1(S^3 \setminus K; \Lambda) & \xrightarrow{f_*} & H_1(\Sigma^q(K)) \\ \downarrow i_{\pm} & & \downarrow j_{\pm} \\ H_1(D_{\pm}^4 \setminus D_{\pm}; \Lambda) & \xrightarrow{g_*} & H_1(W_{\pm}). \end{array}$$

In the above diagram, j_{\pm} are homomorphisms induced from inclusions, and f_* and g_* are the canonical surjections sending a Λ -module to its quotient by $t^q - 1$.

Let $G_{\pm} := f_*(M_{\pm})$ and $G_{\pm}^i := f_*(M_{\pm}^i)$ for $i = 1, 2$. Since we have $H_1(S^3 \setminus K; \Lambda) = H_1(S^3 \setminus K_1; \Lambda) \oplus H_1(S^3 \setminus K_2; \Lambda)$ and $H_1(\Sigma^q(K)) = H_1(\Sigma^q(K_1)) \oplus H_1(\Sigma^q(K_2))$ as Λ -modules and f_* preserves the direct sums, we have $G_{\pm}^i \subset H_1(\Sigma^q(K_i))$ for $i = 1, 2$. Since M_{\pm}^i are metabolizers for the Blanchfield form on $H_1(S^3 \setminus K_i; \Lambda)$ for $i = 1, 2$, by Proposition 2.3, G_{\pm}^i are metabolizers for the linking form $H_1(\Sigma^q(K_i) \times H_1(\Sigma^q(K_i)) \rightarrow \mathbb{Q}/\mathbb{Z}$ for $i = 1, 2$. Similarly, G is a metabolizer for the linking form on $H_1(\Sigma^q(K))$. Moreover, we have $H_1(\Sigma^q(K_i)) = G_+^i \oplus G_-^i$ for $i = 1, 2$.

Now we show that $\bar{d}(\Sigma^q(K_1), \mathfrak{s}_0 + a) = 0$ for all $a \in G_+^1 \cup G_-^1$. Let $a \in G_+^1$. Then $(a, 0) \in G_+^1 \oplus G_+^2 = G_+ \subset H_1(\Sigma^q(K))$. Since $j_+(G_+) = (j_+ \circ f_*)(M_+) = (g_* \circ i_+)(M_+) = g_*(0) = 0$, we have $G_+ \subset \text{Ker}(j_+)$. Furthermore, since both G_+ and $\text{Ker}(j_+)$ are metabolizers for the linking form on $H_1(\Sigma^q(K))$, we have $|G_+| = |\text{Ker}(j_+)|$ and hence $G_+ = \text{Ker}(j_+)$. Therefore, since D_+ is a slice disk for K , $d(\Sigma^q(K), \mathfrak{s}_0 + (a, 0)) = 0$ by Theorem 2.4. Again,

since K is slice, by Theorem 2.4 we have $d(\Sigma^q(K), \mathfrak{s}_0) = 0$. Therefore,

$$\begin{aligned}
0 &= d(\Sigma^q(K), \mathfrak{s}_0 + (a, 0)) \\
&= d(\Sigma^q(K), \mathfrak{s}_0 + (a, 0)) - d(\Sigma^q(K), \mathfrak{s}_0) \\
&= d(\Sigma^q(K_1), \mathfrak{s}_0 + a) + d(\Sigma^q(K_2), \mathfrak{s}_0) - \\
&\quad (d(\Sigma^q(K_1), \mathfrak{s}_0) + d(\Sigma^q(K_2), \mathfrak{s}_0)) \\
&= d(\Sigma^q(K_1), \mathfrak{s}_0 + a) - d(\Sigma^q(K_1), \mathfrak{s}_0) \\
&= \bar{d}(\Sigma^q(K_1), \mathfrak{s}_0 + a).
\end{aligned}$$

Similarly, $\bar{d}(\Sigma^q(K_1), \mathfrak{s}_0 + a) = 0$ for all $a \in G_-^1$. Therefore K_1 has doubly vanishing d -invariants. Using similar arguments, one can show that K_2 also has doubly vanishing d -invariants. \square

Next, we give a proof of Lemma 3.1.

Proof of Lemma 3.1. The proof is almost the same as the one of [KK08, Theorem 3.1]. The inclusion $M_+^1 \oplus M_+^2 \subset M_+$ is obvious, and we show that $M_+ \subset M_+^1 \oplus M_+^2$. For brevity, let $\Delta_i := \Delta_{K_i}(t)$, the Alexander polynomial of K_i , for $i = 1, 2$. Since Δ_1 and Δ_2 are coprime in $\mathbb{Q}[t^{\pm 1}]$, there exist $\bar{f}_1, \bar{f}_2 \in \mathbb{Q}[t^{\pm 1}]$ such that $\bar{f}_1 \Delta_1 + \bar{f}_2 \Delta_2 = 1$. Let c be an integer such that $c\bar{f}_1, c\bar{f}_2 \in \Lambda$, and let $f_1 := c\bar{f}_1$ and $f_2 := c\bar{f}_2$. Then we have $f_1 \Delta_1 + f_2 \Delta_2 = c \in \mathbb{Z}$.

Let $z \in M_+$. Then $z = (x, y) \in H_1(S^3 \setminus K_1; \Lambda) \oplus H_1(S^3 \setminus K_2; \Lambda)$. Since Δ_1 annihilates $H_1(S^3 \setminus K_1; \Lambda)$, we have $\Delta_1 x = 0$. Therefore, $cx = (f_1 \Delta_1 + f_2 \Delta_2)x = f_2 \Delta_2 x$. Since Δ_2 annihilates $H_1(S^3 \setminus K_2; \Lambda)$, we have $\Delta_2 y = 0$ and hence $f_2 \Delta_2 y = 0$. Therefore $f_2 \Delta_2 z = (f_2 \Delta_2 x, 0) = (cx, 0)$, and we have $(cx, 0) \in M_+$. Since M_+ is a direct summand of $H_1(S^3 \setminus K; \Lambda)$, which is \mathbb{Z} -torsion free, we have $(x, 0) \in M_+$. Therefore $x \in M_+^1$. Similarly, one can show that $y \in M_+^2$, and now we have $z \in M_+^1 \oplus M_+^2$. It follows that $M_+ = M_+^1 \oplus M_+^2$. Similarly, one can show that $M_- = M_-^1 \oplus M_-^2$.

Now we show that for each $i = 1, 2$, the modules M_{\pm}^i are metabolizers of the Blanchfield form $B\ell_i: H_1(S^3 \setminus K_i; \Lambda) \times H_1(S^3 \setminus K_i; \Lambda) \rightarrow S^{-1}\Lambda/\Lambda$. Let $x_1, x_2 \in M_+^1$. Then $(x_1, 0), (x_2, 0) \in M_+^1 \oplus M_+^2 = M_+$, which is a metabolizer. Therefore $B\ell((x_1, 0), (x_2, 0)) = 0$. Since $B\ell((x_1, 0), (x_2, 0)) = B\ell_1(x_1, x_2) + B\ell_2(0, 0) = B\ell_1(x_1, x_2)$, we have $B\ell_1(x_1, x_2) = 0$. This implies that $M_+^1 \subset (M_+^1)^{\perp}$.

Conversely, let $x_1 \in (M_+^1)^{\perp}$. Then, for every $(x_2, y) \in M_+^1 \oplus M_+^2 = M_+$, we have $B\ell((x_1, 0), (x_2, y)) = B\ell_1(x_1, x_2) + B\ell_2(0, y) = 0 + 0 = 0$. Therefore $(x_1, 0) = (M_+)^{\perp} = M_+$. Since $M_+ = M_+^1 \oplus M_+^2$, we have $x_1 \in M_+^1$. Therefore $(M_+^1)^{\perp} \subset M_+^1$, and consequently we have $M_+^1 = (M_+^1)^{\perp}$ and M_+^1 is a metabolizer for $B\ell_1$.

Similarly, we can show that M_+^2, M_-^1 , and M_-^2 are also metabolizers. \square

We finally prove Theorem 1.3.

Proof of Theorem 1.3. We prove Part (1). The knot K is slice since there is a surgery curve for a slice disk on the left band of the obvious Seifert surface for K . Recall that K is a satellite of T with pattern the 9_{46} knot. It is known that 9_{46} is doubly slice, and since $\Delta_T(t) = 1$, T is topologically doubly slice by Freedman's work. Now K is a satellite of a topologically doubly slice knot whose pattern is doubly slice, and therefore K is topologically doubly slice (see [Mei15, Proposition 3.4]).

We show that K is not doubly slice. The needed computation is already done in [CHH13]. Our knot K is the same as the knot $K = R(J, T)$ in [CHH13, Figure 8.1] with the choice $J = U$, the unknot. As computed in [CHH13, Section 8], for the 3-fold

branched cyclic cover of S^3 over K , one can show that $H_1(\Sigma^3(K)) \cong \mathbb{Z}_7\langle x_1 \rangle \oplus \mathbb{Z}_7\langle y_1 \rangle$ and the linking form on $H_1(\Sigma^3(K))$ has only two metabolizers $\langle x_1 \rangle$ and $\langle y_1 \rangle$. In the proof of Lemma 8.2 in [CHH13], it is assumed that $K = R(U, T)$, which is the same as our K , and it is computed that $d(\Sigma^3(K), \mathfrak{s}_0 + 4x_2) \leq -\frac{3}{2}$ for the element x_2 such that $4x_2 = x_1$. Since the linking form on $H_1(\Sigma^3(K))$ has only two metabolizers $\langle x_1 \rangle$ and $\langle y_1 \rangle$, it follows that K does not have doubly vanishing d -invariants. In particular, by Theorem 2.5, K is not doubly slice.

We prove Part (2). Since each K_{p_i} has the same Alexander polynomial with $T_{2,p_i} \# T_{2,-p_i}$, we have $\Delta_{K_{p_i}}(t) = \phi_{2p_i}^2$ where ϕ_q denotes the q -th cyclotomic polynomial. Also note that $\Delta_K(t) = (2t - 1)(t - 2)$. Therefore all nK and $n_i K_{p_i}$ have mutually coprime Alexander polynomials.

Suppose $n = 1$. By Part (1), the knot K does not have doubly vanishing d -invariants. Therefore by Theorem 1.2 the knot $K \# (\#_{i=1}^m n_i K_{p_i})$ is not doubly slice.

Suppose $n_i = 1$ for some i . By rearranging p_i , we may assume $n_1 = 1$. It was shown in [Mei15, Corollary 5.2] that K_{p_i} does not have doubly vanishing d -invariants. Again by Theorem 1.2 the knot $nK \# (\#_{i=1}^m n_i K_{p_i}) = K_{p_1} \# nK \# (\#_{i=2}^m n_i K_{p_i})$ is not doubly slice. \square

References

- [Bao15] Yuanyuan Bao, *Polynomial splittings of Ozsváth and Szabó's d -invariant*, Topology Proc. **46** (2015), 309–322.
- [CHH13] Tim D. Cochran, Shelly Harvey, and Peter Horn, *Filtering smooth concordance classes of topologically slice knots*, Geom. Topol. **17** (2013), no. 4, 2103–2162.
- [CHL11] Tim D. Cochran, Shelly Harvey, and Constance Leidy, *Primary decomposition and the fractal nature of knot concordance*, Math. Ann. **351** (2011), no. 2, 443–508.
- [COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and L^2 -signatures*, Ann. of Math. (2) **157** (2003), no. 2, 433–519.
- [Fri03] Stefan Klaus Friedl, *Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants*, ProQuest LLC, Ann Arbor, MI, 2003, Thesis (Ph.D.)—Brandeis University.
- [GRS08] J. Elisenda Grigsby, Daniel Ruberman, and Sašo Strle, *Knot concordance and Heegaard Floer homology invariants in branched covers*, Geom. Topol. **12** (2008), no. 4, 2249–2275.
- [HLR12] Matthew Hedden, Charles Livingston, and Daniel Ruberman, *Topologically slice knots with nontrivial Alexander polynomial*, Adv. Math. **231** (2012), no. 2, 913–939.
- [Jab12] Stanislav Jabuka, *Concordance invariants from higher order covers*, Topology Appl. **159** (2012), no. 10–11, 2694–2710.
- [Kim05] Se-Goo Kim, *Polynomial splittings of Casson-Gordon invariants*, Math. Proc. Cambridge Philos. Soc. **138** (2005), no. 1, 59–78.
- [Kim09] ———, *Alexander polynomials and orders of homology groups of branched covers of knots*, J. Knot Theory Ramifications **18** (2009), no. 7, 973–984.
- [KK08] Se-Goo Kim and Taehee Kim, *Polynomial splittings of metabelian von Neumann rho-invariants of knots*, Proc. Amer. Math. Soc. **136** (2008), no. 11, 4079–4087.
- [KK14] ———, *Splittings of von Neumann rho-invariants of knots*, J. Lond. Math. Soc. (2) **89** (2014), no. 3, 797–816.
- [Lev69] Jerome P. Levine, *Invariants of knot cobordism*, Invent. Math. **8** (1969), 98–110; addendum, ibid. **8** (1969), 355.
- [Mei15] Jeffrey Meier, *Distinguishing topologically and smoothly doubly slice knots*, J. Topol. **8** (2015), no. 2, 315–351.
- [OS03] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261.

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